

PROOF LENGTHS FOR INSTANCES OF THE PARIS-HARRINGTON PRINCIPLE

ANTON FREUND

ABSTRACT. As Paris and Harrington have famously shown, Peano Arithmetic does not prove that for all numbers k, m, n there is an N which satisfies the statement $\text{PH}(k, m, n, N)$: For any k -colouring of its n -element subsets the set $\{0, \dots, N-1\}$ has a large homogeneous subset of size $\geq m$. At the same time very weak theories can establish the Σ_1 -statement $\exists_N \text{PH}(\bar{k}, \bar{m}, \bar{n}, N)$ for any fixed parameters k, m, n . Which theory, then, does it take to formalize *natural* proofs of these instances? It is known that $\forall_m \exists_N \text{PH}(\bar{k}, m, \bar{n}, N)$ has a natural and short proof (relative to n and k) by Σ_{n-1} -induction. In contrast, we show that there is an elementary function e such that any proof of $\exists_N \text{PH}(e(n), n+1, \bar{n}, N)$ by Σ_{n-2} -induction is ridiculously long.

We recall some terminology from [PH77]: For a set X and a natural number n we write $[X]^n$ for the collection of subsets of X with precisely n elements. Given a function f with domain $[X]^n$, a subset Y of X is called homogeneous for f if the restriction of f to the set $[Y]^n$ is constant. A non-empty subset of \mathbb{N} is called large if its cardinality is at least as big as its minimal element. Where the context suggests it we use N to denote the set $\{0, \dots, N-1\}$. Then the Paris-Harrington Principle, or Strengthened Finite Ramsey Theorem, expresses that for all natural numbers k, m, n there is an N such that the following statement holds:

$$\text{PH}(k, m, n, N) \quad \equiv \quad \text{“for any function } [N]^n \rightarrow k \text{ the set } N \text{ has a large homogeneous subset with at least } m \text{ elements”}$$

Using the methods presented in [HP93, Section I.1(b)] it is easy to formalize the statement $\text{PH}(k, m, n, N)$ in the language of first order arithmetic, as a formula that is Δ_1 in the theory $\mathbf{I}\Sigma_1$ of Σ_1 -induction. The celebrated result of [PH77] says that the formula $\forall_{k,m,n} \exists_N \text{PH}(n, m, k, N)$ is true but unprovable in Peano Arithmetic. As is well-known, any true Σ_1 -formula in the language of first-order arithmetic can be proved in a theory as weak as Robinson Arithmetic. It is thus pointless to ask whether a Σ_1 -sentence is provable in a sound arithmetical theory, in contrast to the situation for Π_1 -sentences (cf. Gödel’s Theorems) and Π_2 -sentences (provably total functions). What we can sensibly ask is whether a Σ_1 -sentence has a proof with some additional property. The present paper explores this question for instances $\exists_N \text{PH}(\bar{k}, \bar{m}, \bar{n}, N)$ of the Paris-Harrington Principle. Our principal result states that, for some elementary function e , the following holds:

- (1) For sufficiently large n , no proof of the formula $\exists_N \text{PH}(\overline{e(n)}, \overline{n+1}, \bar{n}, N)$ in the theory $\mathbf{I}\Sigma_{n-2}$ can have Gödel number smaller than $F_{\varepsilon_0}(n-3)$.

If we replace $\mathbf{I}\Sigma_{n-2}$ by $\mathbf{I}\Sigma_{n-3}$ (and $F_{\varepsilon_0}(n-3)$ by $F_{\varepsilon_0}(n-4)$) then we can take the constant function $e(n) = 8$. It is open whether we can make e constant and keep the stronger fragment $\mathbf{I}\Sigma_{n-2}$.

Recall that F_{ε_0} is the function at stage ε_0 of the fast-growing hierarchy. Ketonen

and Solovay in [KS81] have related it to the function that maps (k, m, n) to the smallest witness N which makes the statement $\text{PH}(\bar{k}, \bar{m}, \bar{n}, \bar{N})$ true. It is a classical result of proof theory that F_{ε_0} eventually dominates any provably total function of Peano Arithmetic (see [Kre52, Wai70, Sch71]). Similar to (1) we will show that the Σ_1 -formula $\exists_y F_{\varepsilon_0}(\bar{n}) = y$ has no short proof in the theory \mathbf{IS}_n .

By [HP93, Theorem II.1.9] the formula $\forall_m \exists_N \text{PH}(\bar{k}, m, \bar{n}, N)$ is provable in \mathbf{IS}_{n-1} , for each fixed $n \geq 2$ and k . The proofs of these instances formalize perfectly natural mathematical arguments. According to [HP93, Section II.2(c)] they can be constructed in the meta-theory \mathbf{IS}_1 . Since all provably total functions of \mathbf{IS}_1 are primitive recursive, this complements (1) by the following statement:

- (2) There is a primitive recursive function which maps (k, n) with $n \geq 2$ to a proof of the formula $\forall_m \exists_N \text{PH}(\bar{k}, m, \bar{n}, N)$ in the theory \mathbf{IS}_{n-1} .

Similarly, a primitive recursive construction yields proofs of $\exists_y F_{\varepsilon_0}(\bar{n}) = y$ in the theories \mathbf{IS}_{n+1} : In view of $F_{\varepsilon_0}(x) \simeq F_{\omega_{x+1}}(x) = F_{\omega_x^{x+1}}(x)$ it suffices to prove the statements “ $F_{\omega_n^{n+1}}$ is total”. This is done by Π_2 -induction up to ω_n^{n+1} , which is available in \mathbf{IS}_{n+1} by Gentzen’s classical construction (cf. [FW98, Theorem 4.11]).

We argue that (1) is not only a result about proof length, but also about the existence of natural proofs: Observe first that we are concerned with sequences of proofs for a sequence of parametrized statements, rather than with a single proof of a single statement. Indeed, in a *proof idea* one seems to grasp general features of the situation which allow to deduce the desired statement. Insofar as these features are general they should apply to a class of similar situations, leading to a collection of similar results. In the context of arithmetic these results will often form a sequence $(A_n)_{n \in \mathbb{N}}$ of formulas, indexed in a natural way. A proof idea would then involve instructions to construct a corresponding sequence $(p_n)_{n \in \mathbb{N}}$ of proofs. It is the role of the proofs p_n to guarantee that the formulas A_n are true; the statement “the given instructions produce formally correct proofs p_n of the statements A_n ” should, on the other hand, be justified by fairly elementary means. Since elementary means cannot prove the totality of functions with a high growth rate this implies that the function mapping n to (a code of) the proof p_n cannot grow too fast. In this sense (1) shows that the Paris-Harrington Principle for arity n and $e(n)$ colours has no natural proof in the theory \mathbf{IS}_{n-2} . The author sees no formal condition which, on the positive side, would ensure that a sequence of proofs is natural. On an informal level, the construction which establishes [HP93, Theorem II.1.9] appears to provide natural proofs of $\forall_m \exists_N \text{PH}(\bar{k}, m, \bar{n}, N)$ in the theory \mathbf{IS}_{n-1} .

Let us summarize the content of the different sections: In Section 1 we show how the analysis of reflection leads to lower bounds on proof sizes. Given a theory \mathbf{T} , the uniform reflection principle for the formula $\exists_y \varphi(x, y)$ expresses that “for all p and n there is an N such that if p is a \mathbf{T} -proof of $\exists_y \varphi(\bar{n}, y)$ then $\varphi(\bar{n}, \bar{N})$ is true”. If we have a bound on the provably total functions of reflection then we know that the witness N cannot be too much bigger than the code of the proof p . Vice versa p cannot be too small if $\exists_y \varphi(\bar{n}, y)$ has only large witnesses. We suppose that this line of argumentation is known (it occurs e.g. in [HMP93]), but the author knows of no article that would systematically apply it to strong theories.

The method just described applies to sequences of proofs in a single theory \mathbf{T} , while

statement (1) is concerned with a sequence of proofs that may contain axioms from increasingly strong theories. This discrepancy is resolved in Section 2: We consider a notion of “slow proof” in Peano Arithmetic, deduced from the slow consistency statement introduced by Sy-David Friedman, Rathjen and Weiermann in [FRW13]. The idea is to penalize complex induction axioms by a drastic increase in proof size. This generates an interplay between proof length and the use of induction. At the same time it makes the construction of proofs more difficult, thus weakening the reflection and consistency statement. We can then apply the method of Section 1 to show that any slow **PA**-proof of $\exists_N \text{PH}(\overline{e(n+2)}, \overline{n+3}, \overline{n+2}, N)$ must be long. Claim (1) will easily follow.

The results of Section 2 rely on certain bounds on the provably total functions of slow reflection. The proof of these bounds follows in Section 3.

Our results depend on a choice of Gödel numbers for proofs but they are quite robust: By Remark 1.4 a primitive recursive change in the coding would merely change the precise meaning of “sufficiently large” in (1). We do require that the relation “ p codes a proof of the statement with Gödel number φ in the theory \mathbf{IS}_n ” is defined by a formula $\text{Proof}_{\mathbf{IS}_n}(p, \varphi)$ with parameters n, p and φ . This formula must be Δ_1 in the theory \mathbf{IS}_1 , and it has to be equivalent to one of the common formalisms (e.g. sequent calculus). Similarly, our results are not very sensitive to the precise formula which we choose as a formalization of $\text{PH}(k, m, n, N)$.

1. BOUNDING PROOF SIZES VIA REFLECTION PRINCIPLES

In this section we show how bounds on the provably total functions of uniform Σ_1 -reflection lead to lower bounds on the sizes of proofs. To formulate the reflection principle we will need a Σ_1 -formula $\text{True}_{\Sigma_1}(\varphi)$ that defines truth for Σ_1 -formulas (in the large sense, i.e. the formula may start with several existential quantifiers). The theory \mathbf{IS}_1 should be able to prove Tarski’s truth conditions (as guaranteed by [HP93, Theorem I.1.75]). With respect to the truth predicate we must develop the theory in some generality:

Definition 1.1. A proof predicate is a Π_1 -formula $\text{Proof}(p, \varphi)$ in the language of first-order arithmetic, with only the variables p and φ free. Given a proof predicate we have the associated Σ_1 -reflection principle

$$\text{RFN}_{\Sigma_1} := \forall \varphi (\text{“}\varphi \text{ is a closed } \Sigma_1\text{-formula”} \wedge \exists p \text{Proof}(p, \varphi) \rightarrow \text{True}_{\Sigma_1}(\varphi)).$$

For a natural number p and a formula φ with Gödel number $\ulcorner \varphi \urcorner$ we say that “ p is a proof of φ ” if the formula $\text{Proof}(\overline{p}, \ulcorner \varphi \urcorner)$ is true in the standard model.

The following observation is easy but crucial:

Lemma 1.2. *Let $\text{Proof}(p, \varphi)$ be a proof predicate, and let \mathbf{T} be a sound extension of \mathbf{IS}_1 that proves the Σ_1 -reflection principle associated with $\text{Proof}(p, \varphi)$. For any Σ_1 -formula $\psi(x, y)$ there is a \mathbf{T} -provably total function $g : \mathbb{N}^2 \rightarrow \mathbb{N}$ such that $\psi(\overline{n}, \overline{g(p, n)})$ is true whenever p is a proof of $\exists y \psi(\overline{n}, y)$.*

Note that, since \mathbf{T} must be sound, the lemma can only be applied to proof predicates which are themselves sound for Σ_1 -formulas.

Proof. Since the theory \mathbf{T} extends \mathbf{IS}_1 it is strong enough to handle Feferman’s dot notation, and it proves the “It’s snowing”-Lemma (see [HP93, Corollary I.1.76]).

Combining this with the reflection principle for $\text{Proof}(p, x)$ we obtain

$$\mathbf{T} \vdash \forall x (\exists p \text{Proof}(p, \ulcorner \exists y \psi(\dot{x}, y) \urcorner) \rightarrow \exists y \psi(x, y)).$$

Prefixing quantifiers transforms this into

$$\mathbf{T} \vdash \forall_{p,x} \exists y (\text{Proof}(p, \ulcorner \exists y \psi(\dot{x}, y) \urcorner) \rightarrow \psi(x, y)).$$

We remark that it is only mildly non-constructive to prefix the existential quantifier in the consequent: A computation of the witness y will use the proof p but rather not the computational content of the statement $\text{Proof}(p, \ulcorner \exists y \psi(\dot{x}, y) \urcorner)$. In any case the formula $\text{Proof}(p, \ulcorner \exists y \psi(\dot{x}, y) \urcorner) \rightarrow \psi(x, y)$ is Σ_1 in \mathbf{IS}_1 . As we have seen the theory \mathbf{T} shows that this formula defines a left-total relation. To obtain a single-valued function we apply a standard minimization argument. Note that we cannot simply pick the minimal value for y since this would yield a function with a Δ_2 -graph; instead we simultaneously minimize over y and the witness to the existential quantifier implicit in $\text{Proof}(p, \ulcorner \exists y \psi(\dot{x}, y) \urcorner) \rightarrow \psi(x, y)$. This results in a Σ_1 -formula $\chi(p, x, y)$ such that we have

$$\mathbf{T} \vdash \forall_{p,x,y} (\chi(p, x, y) \rightarrow (\text{Proof}(p, \ulcorner \exists y \psi(\dot{x}, y) \urcorner) \rightarrow \psi(x, y)))$$

and $\mathbf{T} \vdash \forall_{x,p} \exists!_y \chi(p, x, y)$. Since \mathbf{T} is sound the formula $\chi(p, x, y)$ does indeed define a \mathbf{T} -provably total function $g : \mathbb{N}^2 \rightarrow \mathbb{N}$, which satisfies $\mathbb{N} \models \chi(\bar{p}, \bar{n}, g(p, n))$ for all natural numbers p and n . By the above we also have

$$\mathbb{N} \models \text{Proof}(\bar{p}, \ulcorner \exists y \psi(\bar{n}, y) \urcorner) \rightarrow \psi(\bar{n}, g(p, n)) \quad \text{for all } p, n \in \mathbb{N}.$$

Lifting the implication to the meta-language gives the desired claim. \square

We can deduce the promised lower bound on proof sizes:

Proposition 1.3. *Let $\text{Proof}(p, \varphi)$ be a proof predicate, and let \mathbf{T} be a sound extension of \mathbf{IS}_1 that proves the Σ_1 -reflection principle for $\text{Proof}(p, \varphi)$. Consider a Σ_1 -formula $\psi(x, y)$ and define a function $F_\psi : \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$ by setting*

$$F_\psi(n) := \begin{cases} m & \text{if } m \text{ is the least number for which } \psi(\bar{n}, \bar{m}) \text{ is true,} \\ \infty & \text{if } \exists y \psi(\bar{n}, y) \text{ is false.} \end{cases}$$

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function with $f(n) \geq n$ and such that, whenever g is \mathbf{T} -provably total, the function $g \circ f$ is eventually dominated by F_ψ (considering ∞ as bigger than any natural number). Then there is a bound N such that we have

$$p > f(n) \quad \text{whenever } p \text{ is a proof of } \exists y \psi(\bar{n}, y) \text{ with } n \geq N.$$

To avoid misunderstanding, we stress that the notion of proof in the last line of the proposition is induced by the truth predicate in the first line, via Definition 1.1.

Proof. Let $g : \mathbb{N}^2 \rightarrow \mathbb{N}$ be the function provided by Lemma 1.2. We can make g monotone in both arguments: First define $g_0 : \mathbb{N}^2 \rightarrow \mathbb{N}$ by the primitive recursion

$$\begin{aligned} g_0(p, 0) &:= g(p, 0), \\ g_0(p, n+1) &:= \max\{g(p, n+1), g_0(p, n)\}. \end{aligned}$$

This yields $g_0(p, n) \geq g(p, n)$ for all numbers p and n , as well as $g_0(p, n) \leq g_0(p, n')$ whenever we have $n \leq n'$. Now define $g_1 : \mathbb{N}^2 \rightarrow \mathbb{N}$ by setting

$$\begin{aligned} g_1(0, n) &:= g_0(0, n), \\ g_1(p+1, n) &:= \max\{g_0(p+1, n), g_1(p, n)\}. \end{aligned}$$

It is obvious that we have $g_1(p, n) \geq g_0(p, n) \geq g(p, n)$ for all numbers p and n , and that $p \leq p'$ implies $g_1(p, n) \leq g_1(p', n)$. By induction on p one can also show that $g_1(p, n) \leq g_1(p, n')$ holds whenever we have $n \leq n'$. Lemma 1.2 implies that we have

$$F_\psi(n) \leq g_1(p, n) \quad \text{whenever } p \text{ is a proof of } \exists_y \psi(\bar{n}, y).$$

Since the theory \mathbf{T} extends \mathbf{IS}_1 its provably total functions are closed under primitive recursion, by [HP93, Theorem I.1.54]. Thus g_1 is still \mathbf{T} -provably total. We define another \mathbf{T} -provably total function $g^\Delta : \mathbb{N} \rightarrow \mathbb{N}$, diagonalizing over g_1 , as

$$g^\Delta(p) := g_1(p, p) + 1.$$

By assumption there is a bound N such that we have

$$(g^\Delta \circ f)(n) \leq F_\psi(n) \quad \text{for all } n \geq N.$$

Let us show that the same bound N satisfies the claim of the proposition: Consider an arbitrary $n \geq N$ and assume that p is a proof of the formula $\exists_y \psi(\bar{n}, y)$. Aiming at a contradiction we assume $p \leq f(n)$. Then we have

$$F_\psi(n) \leq g_1(p, n) \leq g_1(f(n), f(n)) < (g^\Delta \circ f)(n) \leq F_\psi(n),$$

which is indeed absurd. \square

It is a nice property of the proposition that the bounds it establishes are invariant under basic transformations of proofs:

Remark 1.4. If f satisfies the conditions of the proposition and h is \mathbf{T} -provably total (e.g. primitive recursive) with $h(p) \geq p$ then $h \circ f$ satisfied these conditions as well. Thus proofs of $\exists_y \psi(\bar{n}, y)$ will even be bigger than $h(f(n))$ for all n above some (possibly increased) bound.

This is useful because it allows us to preprocess proofs: Consider a modified notion proof' and a sequence of formulas φ_n , not necessarily of the form $\varphi(\bar{n})$ and not necessarily in the syntactic class Σ_1 . Assume that there is a Σ_1 -formula $\psi(x, y)$ and a primitive recursive function h which transforms any proof' of φ_n into (an upper bound for) a proof of $\exists_y \psi(\bar{n}, y)$. Possibly increasing h we can assume that h is monotone and satisfies $h(p) \geq p$. Using the proposition we may be able to show that $p > h(f(n))$ holds whenever p is a proof of $\exists_y \psi(\bar{n}, y)$, with n sufficiently large. We want to deduce $q > f(n)$ where q is a proof' of φ_n . Indeed, $q \leq f(n)$ would imply $h(q) \leq h(f(n))$. This would mean that there exists a proof of $\exists_y \psi(\bar{n}, y)$ below $h(f(n))$, which we have seen to be false. The proof of Lemma 2.8 contains a detailed application of this argument.

To conclude this section we illustrate what a simple application of the proposition can yield. Adapting the notation from [KS81] we have

$$\sigma(n, k) = \min\{N \mid \text{PH}(\bar{k}, \overline{n+1}, \bar{n}, \bar{N}) \text{ is true}\},$$

i.e. the number $\sigma(n, k)$ is the smallest witness for the Paris-Harrington Principle with arity n and k colours. We know from [PH77, Theorem 3.2] that the function $n \mapsto \sigma(n, n)$ eventually dominates any provably total function of Peano Arithmetic. The following result on proof sizes is considerably weaker than (1), insofar as it speaks about fixed fragments of Peano Arithmetic.

Corollary 1.5. *For any number k the (total) function*

$$n \mapsto \text{“the smallest Gödel number of a proof of the } \Sigma_1\text{-formula } \exists_N PH(\bar{n}, \bar{n} + 1, \bar{n}, N) \text{ by } \Sigma_k\text{-induction”}$$

eventually dominates any provably total function of Peano Arithmetic.

Proof. Let f be an arbitrary **PA**-provably total function. Assume that $f(n) \geq n$ holds for all n , possibly after replacing f by the function $n \mapsto \max\{f(n), n\}$. We apply Proposition 1.3 to the usual proof predicate $\text{Proof}_{\mathbf{I}\Sigma_k}(p, \varphi)$ for the theory of Σ_k -induction (or rather to a Π_1 -formula that is equivalent to $\text{Proof}_{\mathbf{I}\Sigma_k}(p, \varphi)$ over $\mathbf{I}\Sigma_1$), to the theory $\mathbf{T} = \mathbf{PA}$, to the formula $\psi(x, y) \equiv PH(x, x + 1, x, y)$, and to the function f . Then $n \mapsto \sigma(n, n)$ is the function F_ψ of Proposition 1.3. The assumptions of the proposition are satisfied: It is well known that Peano arithmetic proves uniform Σ_1 -reflection for the theory $\mathbf{I}\Sigma_k$ (see e.g. [HP93, Corollary I.4.34]). For any **PA**-provably total function g the composition $g \circ f$ is **PA**-provably total as well, and thus indeed dominated by $n \mapsto \sigma(n, n)$. The result of Proposition 1.3 is nothing but the claim of the corollary. \square

The bound of the corollary is reasonably accurate, in the sense that the function computing the minimal proofs is not much faster than the provably total functions of Peano Arithmetic: Recall that $PH(k, m, n, N)$ is Δ_1 in $\mathbf{I}\Sigma_1$. Thus not only $\sigma(n, n)$ itself but the witnesses to all unbounded quantifiers of the Σ_1 -formula $\exists_N PH(\bar{n}, \bar{n} + 1, \bar{n}, N)$ are bounded by a primitive recursive function in n and $\sigma(n, n)$. Furthermore, the Σ_1 -completeness theorem is established by a primitive recursive construction of proofs. Thus there is a primitive recursive function $h : \mathbb{N}^2 \rightarrow \mathbb{N}$ such that $h(n, \sigma(n, n))$ is the Gödel number of a proof of $\exists_N PH(\bar{n}, \bar{n} + 1, \bar{n}, N)$ in the theory $\mathbf{I}\Sigma_0$.

2. NO SHORT PROOFS FOR INSTANCES OF THE PARIS-HARRINGTON PRINCIPLE

In this section we refine Corollary 1.5 by varying k alongside with n . On first sight it may seem astonishing that Proposition 1.3, which only deals with one proof predicate at a time, can be used to this effect. We will see, however, that a single proof predicate can inform us about proofs in various theories: The slow **PA**-proofs that we will introduce penalize the use of complex induction axioms by a drastic increase in proof length, thus creating an interplay between proof length and the amount of induction used in the proof.

Before we can define the notion of a slow proof we need some preliminaries on ordinal notations and the fast-growing hierarchy of functions. Ordinal notations are required for the ordinals below ε_0 , the smallest fixed point of the function $\alpha \mapsto \omega^\alpha$. As usual they will be based on the Cantor normal form

$$\alpha = \omega^{\alpha_1} \cdot n_1 + \cdots + \omega^{\alpha_k} \cdot n_k \quad \text{with } k \in \mathbb{N}, n_i \in \mathbb{N} \setminus \{0\} \text{ and } \alpha_1 > \cdots > \alpha_k.$$

Crucially, $\alpha < \varepsilon_0$ implies $\alpha_1 < \alpha$ so that the Cantor normal form inductively yields finite term notations. Basic ordinal arithmetic can be translated into syntactic operations on these terms. The operations are sufficiently elementary to make ordinal arithmetic available in the theory $\mathbf{I}\Sigma_1$, after arithmetization of the finite term syntax. In fact, Sommer in [Som95, Sections 2 and 3] shows that theories much weaker than $\mathbf{I}\Sigma_1$ suffice if one encodes the terms efficiently. In this paper we are not interested in very weak theories, but it is nevertheless convenient to adopt the encoding of Sommer: This allows us to use his Δ_0 -definition of the functions in

the fast-growing hierarchy.

We remark that the ordinal arithmetic of [Som95] includes fundamental sequences: The fundamental sequence $(\{\alpha\}(n))_{n \in \mathbb{N}}$ of a limit ordinal α is a strictly increasing sequence of ordinals with supremum α . Precisely, any limit ordinal α can uniquely be written as $\alpha = \beta + \omega^\gamma \cdot (k+1)$ where $\gamma > 0$ is the smallest exponent of the Cantor normal form of α , and β contains the larger summands. We then have

$$\begin{aligned} \{\beta + \omega^\gamma \cdot (k+1)\}(n) &= \beta + \omega^\gamma \cdot k + \omega^\delta \cdot (n+1) & \text{if } \gamma = \delta + 1, \\ \{\beta + \omega^\gamma \cdot (k+1)\}(n) &= \beta + \omega^\gamma \cdot k + \omega^{\{\gamma\}(n)} & \text{if } \gamma \text{ is a limit.} \end{aligned}$$

For zero and successor ordinals one sets $\{0\}(n) := 0$ and $\{\beta + 1\}(0) := \beta$.

Next, consider the “stack of ω ’s”-function defined by the recursion

$$\omega_0^\alpha = \alpha, \quad \omega_{n+1}^\alpha = \omega^{\omega_n^\alpha}.$$

As usual, ω_n abbreviates ω_n^1 . This function is not part of the ordinal arithmetic encoded by Sommer (although it is, of course, part of his meta-theory). Since Sommer does encode the function $\alpha \mapsto \omega^\alpha$ it is immediate to make the function $(n, \alpha) \mapsto \omega_n^\alpha$ (operating on the codes) available in \mathbf{IS}_1 . However, we will need more, namely a Δ_0 -formula defining the graph and explicit bounds on the values of this function. Write $\ulcorner \alpha \urcorner$ for the term notation of α , represented as a list with digits from $\{1, \dots, 4\}$ as in [Som95]. Then ω_n^α is represented by the following concatenation of lists:

$$\ulcorner \omega_n^\alpha \urcorner = \underbrace{\langle 4, \dots, 4 \rangle}_{n \text{ characters } 4} \frown \ulcorner \alpha \urcorner \frown \underbrace{\langle 3, 1, \dots, 3, 1 \rangle}_{n \text{ alternations}}$$

Indeed, with each character 4 we move to the exponent of the leftmost summand of the Cantor normal form, while 3 instructs us to leave the exponent and look at the corresponding coefficient, which in the present case is always 1 (represented by the base two notation of 1, which happens to be the list $\langle 1 \rangle$ itself). Now to verify the relation $\omega_n^\alpha = \beta$ we only have to compare digits in the sequence representations of α and β , and this can be cast into a Δ_0 -formula (see [Som95, Section 2.2]). Using [Som95, Proposition 2.1], which relates the code of a list of digits to its length, we can also establish the following inequality between the codes of α and ω_n^α :

$$(3) \quad \mathbf{IS}_1 \vdash \forall_{n, \alpha} \omega_n^\alpha \leq 4^{3n+1} \cdot (\alpha + 1).$$

Let us remark that we do not extend the ordinal notation system by a symbol for ε_0 , in order to keep it closed under the usual operations of ordinal arithmetic. By a harmless abuse of notation we will sometimes refer to the “fundamental sequence” of ε_0 , which we define as $\{\varepsilon_0\}(n) := \omega_{n+1}$.

Using fundamental sequences we can define the fast-growing hierarchy of functions indexed by ordinals below and including ε_0 . The definition varies slightly within the literature; our version differs from the classic [Wai70, Sch71] and coincides e.g. with [Som95]:

$$\begin{aligned} F_0(x) &:= x + 1, \\ F_{\alpha+1}(x) &:= F_\alpha^{x+1}(x), \\ F_\lambda(x) &:= F_{\{\lambda\}(x)}(x) \quad \text{for } \lambda \text{ a limit ordinal.} \end{aligned}$$

Here and in the following an exponent to a function symbol denotes the number of times the function is to be iterated. Given an arithmetization of ordinal arithmetic it is easy to define the graph of $(\alpha, x, i) \mapsto F_\alpha^i(x)$ by a Σ_1 -formula in the language of

first-order arithmetic: As described in [Som90, Section 4.1] one can compute $F_\alpha^i(x)$ by simplifying expressions of the form $F_{\alpha_1}^{i_1}(F_{\alpha_2}^{i_2}(\dots(F_{\alpha_k}^{i_k}(z))\dots))$, so one only needs to state the existence of such a computation sequence. What is remarkable is that the size of an (improved) computation sequence can be bounded by a polynomial in the value of $F_\alpha^i(x)$. This is worked out in [Som90, Appendix A] (see also the less detailed [Som95, Section 5.2]) and leads to a Δ_0 -formula $F_\alpha^i(x) = y$ with free variables x, y, α, i which defines the functions F_α for $\alpha \prec \varepsilon_0$, as well as their iterations. By [Som95, Theorem 5.3] the defining equations of the fast-growing hierarchy are provable in $\mathbf{I}\Sigma_1$ (under the assumption that the involved computations terminate, which is of course unprovable in $\mathbf{I}\Sigma_1$). As Sommer only encodes the hierarchy below ε_0 we should show separately that the formula

$$F_{\varepsilon_0}(x) = y \quad \equiv \quad \exists \alpha (\alpha = \omega_{x+1} \wedge F_\alpha(x) = y)$$

is Δ_0 in $\mathbf{I}\Sigma_1$: The only task is to bound the existentially quantified α . By [FRW13, Lemma 2.3, Proposition 2.12] the inequalities

$$F_{\omega_{x+1}}(x) \geq F_\omega(x) \geq F_2(x) = 2^{x+1} \cdot (x+1) - 1 \geq 2^{x+1} \quad \text{for } x \geq 1$$

are provable in $\mathbf{I}\Sigma_1$. Combining this with (3) we obtain

$$(4) \quad \mathbf{I}\Sigma_1 \vdash x \geq 1 \rightarrow (F_{\varepsilon_0}(x) = y \leftrightarrow \exists_{\alpha \leq y^{6.4} \cdot (\ulcorner 1 \urcorner + 1)} (\alpha = \omega_{x+1} \wedge F_\alpha(x) = y)),$$

where $\ulcorner 1 \urcorner$ denotes the code of the ordinal 1.

Writing $\langle \cdot, \cdot \rangle$ for the Cantor pairing function with projections $\pi_1(\cdot), \pi_2(\cdot)$ we can now define slow proofs in Peano arithmetic. The idea is to penalize the use of complex induction axioms by a drastic increase in proof length, and thus to create an interplay between proof size and the amount of induction used in the proof.

Definition 2.1 (cf. [FRW13]). A pair $\langle q, N \rangle$ is a slow **PA**-proof of a formula φ if there is a number n such that we have $N = F_{\varepsilon_0}(n)$ and such that q codes a (usual) proof of φ in the theory $\mathbf{I}\Sigma_{n+1}$. This notion is defined by the formula

$$\text{Proof}_{\mathbf{PA}}^\diamond(p, \varphi) \equiv \exists x (\text{Proof}_{\mathbf{I}\Sigma_{x+1}}(\pi_1(p), \varphi) \wedge F_{\varepsilon_0}(x) = \pi_2(p)),$$

which is Δ_1 in $\mathbf{I}\Sigma_1$ since by [Som95, Proposition 5.4] the second conjunct implies the bound $x \leq \pi_2(p)$.

For a formula $F(x) = y$ let us abbreviate $\exists_y F(x) = y$ by $F(x) \downarrow$. Also, we write $\text{Pr}_{\mathbf{I}\Sigma_x}(\varphi)$ for the formula $\exists_p \text{Proof}_{\mathbf{I}\Sigma_x}(p, \varphi)$. It is easy to see that the slow provability predicate

$$\text{Pr}_{\mathbf{PA}}^\diamond(\varphi) \equiv \exists_p \text{Proof}_{\mathbf{PA}}^\diamond(p, \varphi)$$

satisfies the equivalence

$$\mathbf{I}\Sigma_1 \vdash \text{Pr}_{\mathbf{PA}}^\diamond(\varphi) \leftrightarrow \exists x (\text{Pr}_{\mathbf{I}\Sigma_{x+1}}(\varphi) \wedge F_{\varepsilon_0}(x) \downarrow).$$

The slow uniform Σ_1 -reflection principle

$$\text{RFN}_{\Sigma_1}^\diamond(\mathbf{PA}) \equiv \forall \varphi (\text{"}\varphi \text{ is a closed } \Sigma_1\text{-formula"} \wedge \text{Pr}_{\mathbf{PA}}^\diamond(\varphi) \rightarrow \text{True}_{\Sigma_1}(\varphi))$$

and the slow consistency statement

$$\text{Con}^\diamond(\mathbf{PA}) \equiv \neg \text{Pr}_{\mathbf{PA}}^\diamond(\overline{\ulcorner 0 = 1 \urcorner})$$

can be characterized as

$$(5) \quad \mathbf{I}\Sigma_1 \vdash \text{RFN}_{\Sigma_1}^\diamond(\mathbf{PA}) \leftrightarrow \forall x (F_{\varepsilon_0}(x) \downarrow \rightarrow \text{RFN}_{\Sigma_1}(\mathbf{I}\Sigma_{x+1}))$$

and

$$\mathbf{I}\Sigma_1 \vdash \text{Con}^\diamond(\mathbf{PA}) \leftrightarrow \forall x (F_{\varepsilon_0}(x) \downarrow \rightarrow \text{Con}(\mathbf{I}\Sigma_{x+1})).$$

As the last equivalence reveals the notion of slow **PA**-proof comes from the article [FRW13] by S.-D. Friedman, Rathjen and Weiermann: These authors introduce the slow consistency statement

$$\text{Con}^*(\mathbf{PA}) \equiv \forall x (F_{\varepsilon_0}(x) \downarrow \rightarrow \text{Con}(\mathbf{IS}_x))$$

and show that we have

$$(6) \quad \mathbf{PA} + \text{Con}^*(\mathbf{PA}) \not\vdash \text{Con}(\mathbf{PA}).$$

It has been pointed out by Michael Rathjen (unpublished) that slow provability satisfies the Gödel-Löb conditions, provably so in \mathbf{IS}_1 . In many respects it thus behaves as the usual provability predicate for Peano Arithmetic. The index shift between our $\text{Con}^\diamond(\mathbf{PA})$ and the formula $\text{Con}^*(\mathbf{PA})$ of [FRW13] has been introduced to improve the bounds on proof sizes that we are about to establish. The central ingredient to these bounds is the following result:

Proposition 2.2. *For any provably total function g of $\mathbf{IS}_1 + \text{RFN}_{\Sigma_1}^\diamond(\mathbf{PA})$ there are numbers k and N such that we have*

$$g(F_{\varepsilon_0}(n \dot{-} 1)) \leq F_{\omega_n+k}(n) \quad \text{for all } n \geq N.$$

The proof of this proposition is rather long, and we defer it to the next section. Here, let us deduce the following larger bound, which we will need for our applications:

Corollary 2.3. *For any provably total function g of $\mathbf{IS}_1 + \text{RFN}_{\Sigma_1}^\diamond(\mathbf{PA})$ there is a number N such that we have*

$$g(F_{\varepsilon_0}(n \dot{-} 1)) \leq F_{\varepsilon_0}(n) \quad \text{for all } n \geq N.$$

Proof. Invoking Proposition 2.2, we may assume that N is bigger than k . Then we have $F_{\omega_n+k}(n) \leq F_{\omega_{n+1}}(n)$ for all $n \geq N$. It remains to show that

$$(7) \quad F_{\omega_{n+1}}(n) \leq F_{\varepsilon_0}(n) \quad \text{holds for all } n \geq 1.$$

To prove this, recall the “step down”-relation of [FRW13, Definition 2.1] (or rather [KS81], with slightly different fundamental sequences): We write $\alpha \rightarrow_n \beta$ to express that there is a sequence $\langle \gamma_0, \dots, \gamma_k \rangle$ with $\gamma_0 = \alpha$, $\gamma_k = \beta$, and $\gamma_{i+1} = \{\gamma_i\}(n)$ for all $i \in \{0, \dots, k-1\}$. Now $F_{\varepsilon_0}(n) = F_{\omega_{n+1}}(n)$ and [FRW13, Lemma 2.3] reduce the inequality (7) to the claim $\omega_{n+1} \rightarrow_n \omega_n + n + 1$. Given $n \geq 1$ we can write $n = n' + 1$. Note that, working in a strong meta-theory, $\alpha \rightarrow_n 0$ is available for any ordinal α . Thus [FRW13, Lemma 2.13, Lemma 2.10] yield

$$\omega_{n+1} = \omega^{\omega_{n'+1}} \rightarrow_n \omega^{\omega_{n'+1}} \rightarrow_n \omega^{\omega_{n'}} \cdot (n+1) = \omega_n \cdot (n+1).$$

Using [FRW13, Lemma 2.8, Proposition 2.12, Lemma 2.7] we also get

$$\omega_n \cdot (n+1) \rightarrow_n \omega_n + \omega_n \rightarrow_n \omega_n + \omega \rightarrow_n \omega_n + n + 1,$$

which completes the proof. \square

Let us remark that the corollary entails

$$\mathbf{IS}_1 + \text{RFN}_{\Sigma_1}^\diamond(\mathbf{PA}) \not\vdash \text{RFN}_{\Sigma_1}(\mathbf{PA}),$$

because the equivalence $\text{RFN}_{\Sigma_1}(\mathbf{PA}) \leftrightarrow F_{\varepsilon_0} \downarrow$ is provable in \mathbf{IS}_1 (see e.g. Proposition 3.1). This result is similar to (6) above, cited from [FRW13]. A more thorough investigation of the slow reflection principle is in planning.

In the rest of this section we show how results about proof sizes in fragments of

Peano Arithmetic can be deduced. First, let us consider proofs of the formulas $F_{\varepsilon_0}(\overline{n}) \downarrow$. Afterwards we will come to the slightly more subtle case of the Paris-Harrington Principle.

Lemma 2.4. *There is a number N such that we have*

$$p > \langle F_{\varepsilon_0}(n \div 1), F_{\varepsilon_0}(n \div 1) \rangle \quad \text{for any slow } \mathbf{PA}\text{-proof } p \text{ of } F_{\varepsilon_0}(\overline{n}) \downarrow, \text{ with } n \geq N.$$

To avoid misunderstanding we recall that $\langle \cdot, \cdot \rangle$ denotes the Cantor pairing.

Proof. We apply Proposition 1.3 to the proof predicate $\text{Proof}_{\mathbf{PA}}^\infty(p, \varphi)$, the theory $\mathbf{T} = \mathbf{IS}_1 + \text{RFN}_{\Sigma_1}^\infty(\mathbf{PA})$, the formula $\psi(x, y) \equiv F_{\varepsilon_0}(x) = y$ (so that F_ψ is the function F_{ε_0}), and the function $n \mapsto \langle F_{\varepsilon_0}(n \div 1), F_{\varepsilon_0}(n \div 1) \rangle$ at the place of f . Let us verify the assumptions of Proposition 1.3: By (5) we have

$$\mathbf{IS}_1 + \text{RFN}_{\Sigma_1}(\mathbf{PA}) \vdash \text{RFN}_{\Sigma_1}^\infty(\mathbf{PA}),$$

where $\text{RFN}_{\Sigma_1}(\mathbf{PA})$ denotes the usual uniform Σ_1 -reflection principle for Peano Arithmetic. This shows that the theory $\mathbf{IS}_1 + \text{RFN}_{\Sigma_1}^\infty(\mathbf{PA})$ is sound. Next, using [KS81, Proposition 2.5] we have

$$n \leq F_{\varepsilon_0}(n \div 1) \leq \langle F_{\varepsilon_0}(n \div 1), F_{\varepsilon_0}(n \div 1) \rangle.$$

Finally, consider an arbitrary function g that is provably total in the theory $\mathbf{IS}_1 + \text{RFN}_{\Sigma_1}^\infty(\mathbf{PA})$. We have to show that there is a number N such that we have

$$g(\langle F_{\varepsilon_0}(n \div 1), F_{\varepsilon_0}(n \div 1) \rangle) \leq F_{\varepsilon_0}(n) \quad \text{for all } n \geq N.$$

This follows from Corollary 2.3, applied not to g itself but rather to the function $m \mapsto g(\langle m, m \rangle)$, which is still provably total in the theory $\mathbf{IS}_1 + \text{RFN}_{\Sigma_1}^\infty(\mathbf{PA})$. Now Proposition 1.3 gives us precisely the claim. \square

It is easy to deduce bounds for proofs in the fragments of Peano Arithmetic:

Theorem 2.5. *There is a number N such that for all $n \geq N$ no proof of the statement $F_{\varepsilon_0}(\overline{n}) \downarrow$ in the theory \mathbf{IS}_n can have code less than or equal to $F_{\varepsilon_0}(n \div 1)$.*

Proof. We can assume that the bound N in Lemma 2.4 is bigger than zero. Let us show that the present result holds with the same bound: Aiming at a contradiction, suppose that $q \leq F_{\varepsilon_0}(n \div 1)$ is an \mathbf{IS}_n -proof of the formula $F_{\varepsilon_0}(\overline{n}) \downarrow$, for some $n \geq N$. By definition $\langle q, F_{\varepsilon_0}(n \div 1) \rangle$ is a slow \mathbf{PA} -proof of $F_{\varepsilon_0}(\overline{n}) \downarrow$. Thus the inequality

$$\langle q, F_{\varepsilon_0}(n \div 1) \rangle \leq \langle F_{\varepsilon_0}(n \div 1), F_{\varepsilon_0}(n \div 1) \rangle$$

contradicts Lemma 2.4. \square

To deduce corresponding results for instances of the Paris-Harrington Principle, we need to link the function $(n, k) \mapsto \sigma(n, k)$ to functions from the fast-growing hierarchy:

Lemma 2.6 ([KS81]). *The following holds:*

(a) *For any number k there is a bound N such that we have*

$$F_{\omega_n+k}(n) \leq \sigma(n+2, 10^{35n^2}) \quad \text{for all } n \geq N.$$

(b) *For any number k there is a bound N such that we have*

$$F_{\omega_n+k}(n) \leq \sigma(n+3, 8) \quad \text{for all } n \geq N.$$

Proof. Recall from the proof of Corollary 2.3 that $F_{\omega_n+k}(n) \leq F_{\varepsilon_0}(n)$ holds for sufficiently large n . Thus it suffices to show that we have

$$F_{\varepsilon_0}(n) \leq \sigma(n+2, 10^{35n^2}) \leq \sigma(n+3, 8) \quad \text{for all } n \geq 15.$$

This is the result of [KS81, Lemma 3.6, Theorem 3.10], except that [KS81] works with a slightly different version of fundamental sequences, setting

$$\{\beta + \omega^\gamma \cdot (k+1)\}(n) = \beta + \omega^\gamma \cdot k + \omega^\delta \cdot n \quad \text{in case } \gamma = \delta + 1.$$

With this definition, descending to the n -th member of the fundamental sequence can introduce a coefficient (bounded by) n . In our case the new coefficients are bounded by $n+1$. The overall bound $\sigma(n+2, 10^{23n^2})$ of [KS81, Lemma 3.6] then increases to our $\sigma(n+2, 10^{35n^2})$.

Let us describe the concrete changes that are necessary (the reader will have to consult [KS81] for context): First, the bound of [KS81, Proposition 2.9] increases from $|T_{k,c,n}| \leq (n+1)_k^c$ to $|T_{k,c,n}| \leq (n+2)_k^c$. At the same time the rather generous bound $|T_{k,c,n}| \leq 2_{k-1}^{(n^{6c})}$ of [KS81, Proposition 2.10] remains valid without change. Thus [KS81, Lemma 3.1] remains valid, and so does [KS81, Lemma 3.2.1]. A small change is required in [KS81, Lemma 3.2.2]: We need to weaken the condition $g(x_0, \dots, x_{n-1}) \leq x_0$ to $g(x_0, \dots, x_{n-1}) \leq x_0 + 1$. It is easy to see that g is then controlled by an $(n+1, 10^5)$ -algebra (instead of an $(n+1, 10^4)$ -algebra). Consequently, [KS81, Lemma 3.2.3] now constructs an $(n+1, 10^{5c})$ -algebra. One can check that [KS81, Lemma 3.4] remains valid in spite of the prior changes: The bound of [KS81, Lemma 3.2.3] is still strong enough for the base case of the proof; in the step, the bound is generous enough to accomodate the fact that G_3 is now an $(n+2, 10^5)$ -algebra. It follows that [KS81, Theorem 3.5] remains unchanged: For $n, k \geq 1$ the function $F_{\omega_n^k}$ is captured by an $(n+2, 10^{n \cdot (12n+2k+8)})$ -algebra. Parallel to [KS81, Lemma 3.6] we can deduce the bound from the beginning of the present proof: We have $\{\omega_{n+1}\} = \omega_n^{n+1}$ and thus $F_{\varepsilon_0}(n) = F_{\omega_{n+1}}(n) = F_{\omega_n^{n+1}}(n)$ (as opposed to $F_{\varepsilon_0}(n) = F_{\omega_n^n}(n)$ in the original [KS81, Lemma 3.6]). Let G_0 be an $(n+2, 10^{14n^2+20n})$ -algebra that captures $F_{\omega_n^{n+1}}$. Let G_1 be an $(n+2, 7)$ -algebra such that $\min(S) \geq 2n+3$ holds whenever S is suitable for G_1 . In view of

$$7 \cdot 10^{14n^2+20n} \leq 10^{14n^2+20n+1} \leq 10^{35n^2} \quad (\text{for } n \geq 1)$$

we can choose an $(n+2, 10^{35n^2})$ -algebra G which simulates G_0 and G_1 . If S is suitable for G then we have

$$\max(S) \geq s_2 > s_1 \geq F_{\omega_n^{n+1}}(s_0) \geq F_{\omega_n^{n+1}}(n) = F_{\varepsilon_0}(n).$$

This means that the restriction

$$G \upharpoonright_{[F_{\varepsilon_0}(n)]^{n+2}}: [F_{\varepsilon_0}(n)]^{n+2} \rightarrow 10^{35n^2}$$

admits no suitable set. Thus we have $F_{\varepsilon_0}(n) < \sigma(n+2, 10^{35n^2})$.

It remains to check $\sigma(n+2, 10^{35n^2}) \leq \sigma(n+3, 8)$. This is parallel to the proof of [KS81, Theorem 3.10]: Observe that we have

$$F_3^{n+1}(n+2) \geq F_3(n) \geq 2^{2^n} \geq 2^{4 \cdot 35n^2} \geq 10^{35n^2} \quad \text{for } n \geq 15.$$

Thus by [KS81, Lemma 3.9] each $(n+2, 10^{35n^2})$ -algebra can be simulated by an $(n+3, 8)$ -algebra, and this implies the claim. Note that the condition $n \geq 15$ could easily be replaced by a smaller bound. \square

This implies the following result, which we will need in our applications:

Corollary 2.7. *For any provably total function g of $\mathbf{IS}_1 + \mathbf{RFN}_{\Sigma_1}^{\circ}(\mathbf{PA})$ the following holds:*

(a) *There is a number N such that we have*

$$g(F_{\varepsilon_0}(n \div 1)) \leq \sigma(n+2, 10^{35n^2}) \quad \text{for all } n \geq N.$$

(b) *There is a number N such that we have*

$$g(F_{\varepsilon_0}(n \div 1)) \leq \sigma(n+3, 8) \quad \text{for all } n \geq N.$$

Proof. This follows from Proposition 2.2 and Lemma 2.6. \square

Similar to Lemma 2.4, slow proofs of certain instances of the Paris-Harrington Principle must be long:

Lemma 2.8. *The following holds:*

- (a) *There is a number K' such that we have $p > \langle F_{\varepsilon_0}(n \div 1), F_{\varepsilon_0}(n \div 1) \rangle$ for any slow \mathbf{PA} -proof p of $\exists_N \text{PH}(\overline{10^{35n^2}}, \overline{n+3}, \overline{n+2}, N)$, with $n \geq K'$.*
- (b) *There is a number K' such that we have $p > \langle F_{\varepsilon_0}(n \div 1), F_{\varepsilon_0}(n \div 1) \rangle$ for any slow \mathbf{PA} -proof p of $\exists_N \text{PH}(8, \overline{n+4}, \overline{n+3}, N)$, with $n \geq K'$.*

Proof. We only show (a). The proof of (b) is similar and somewhat easier. Compared to the proof of Lemma 2.4, the main subtlety is that the formulas

$$\varphi_n := \exists_N \text{PH}(\overline{10^{35n^2}}, \overline{n+3}, \overline{n+2}, N)$$

are not of the form $\varphi(\overline{n})$, i.e. parametrized by the n -th numeral. To make Proposition 1.3 applicable we need to preprocess proofs of these formulas, as sketched in Remark 1.4: Let $e(x) = z$ be a Σ_1 -formula such that we have

$$\mathbb{N} \models e(\overline{n}) = \overline{k} \quad \Leftrightarrow \quad k = 10^{35n^2}$$

and $\mathbf{IS}_1 \vdash \forall_x \exists_z e(x) = z$. In view of the latter, the witnesses to all unbounded quantifiers of the Σ_1 -formula $\exists_z e(\overline{n}) = z$ are bounded by a primitive recursive function in n . By the proof of Σ_1 -completeness there is a primitive recursive function $p_e : \mathbb{N}^2 \rightarrow \mathbb{N}$ such that $p_e(n, k)$ is an \mathbf{IS}_k -proof of $e(\overline{n}) = \overline{10^{35n^2}}$. Next, let $\psi(x, y)$ be a Σ_1 -formula with

$$(8) \quad \mathbf{IS}_1 \vdash \psi(x, y) \leftrightarrow \exists_z (e(x) = z \wedge \text{PH}(z, x+3, x+2, y)).$$

Following Remark 1.4, we need a primitive recursive function $h : \mathbb{N} \rightarrow \mathbb{N}$ which transforms a slow \mathbf{PA} -proof of φ_n into a slow \mathbf{PA} -proof of $\exists_y \psi(\overline{n}, y)$. Let us first construct a primitive recursive function $h' : \mathbb{N}^2 \rightarrow \mathbb{N}$ such that $h'(k, q)$ is an \mathbf{IS}_{k+1} -proof of $\exists_y \psi(\overline{n}, y)$ if q is an \mathbf{IS}_{k+1} -proof of φ_n : Given a proof q as described, we can read off its end formula φ_n and then the number n . Recall that $p_e(n, k+1)$ is an \mathbf{IS}_{k+1} -proof of $e(\overline{n}) = \overline{10^{35n^2}}$. Combining this with q and introducing an existential quantifier yields an \mathbf{IS}_{k+1} -proof of

$$\exists_z (e(\overline{n}) = z \wedge \exists_N \text{PH}(z, \overline{n+3}, \overline{n+2}, N)).$$

It is not unreasonable to assume that $\overline{n+3}$ (resp. $\overline{n+2}$) is the same term as $\overline{n}+3$ (resp. $\overline{n}+2$). Even if not, there are primitive recursive functions which map a pair (k, n) to \mathbf{IS}_{k+1} -proofs of $\overline{n+3} = \overline{n}+3$ and $\overline{n+2} = \overline{n}+2$. We then apply the equality axioms and prefix the existentially quantified N , giving an \mathbf{IS}_{k+1} -proof of

$$\exists_y \exists_z (e(\overline{n}) = z \wedge \text{PH}(z, \overline{n}+3, \overline{n}+2, y)).$$

Invoking the equivalence (8) we get the desired proof $h'(k, q)$ of $\exists_y \psi(\bar{n}, y)$. Now to construct h , assume that $p = \langle q, M \rangle$ is slow **PA**-proof of φ_n . By definition there is an $m \leq M$ such that q is an \mathbf{IS}_{m+1} proof of φ_n and such that we have $F_{\varepsilon_0}(m) = M$. Recall that the relation $F_{\varepsilon_0}(x) = y$ is primitive recursively decidable, and that F_{ε_0} is strongly monotone. Thus we can primitive recursively determine the unique m with the stated property. Now it suffices to set

$$h(p) := \langle h'(m, q), M \rangle.$$

We need to increase h to make it monotone and ensure $h(p) \geq p$. Clearly, the increased function still satisfies the following: If p is a slow **PA**-proof of φ_n then there is a slow **PA**-proof of $\exists_y \psi(\bar{n}, y)$ below $h(p)$.

Now we apply Proposition 1.3 to the proof predicate $\text{Proof}_{\mathbf{PA}}^\circ(p, \varphi)$, the theory $\mathbf{T} = \mathbf{IS}_1 + \text{RFN}_{\Sigma_1}^\circ(\mathbf{PA})$, the Σ_1 -formula $\psi(x, y)$ defined above, and the function $n \mapsto h(\langle F_{\varepsilon_0}(n \div 1), F_{\varepsilon_0}(n \div 1) \rangle)$ at the place of f . In view of (8) we have

$$\mathbb{N} \models \psi(\bar{n}, \bar{m}) \quad \Leftrightarrow \quad \mathbb{N} \models \text{PH}(\overline{10^{35n^2}}, \overline{n+3}, \overline{n+2}, \bar{m}),$$

so that F_ψ is the function $n \mapsto \sigma(n+2, 10^{35n^2})$. Concerning the assumptions of Proposition 1.3, in view of $h(p) \geq p$ (see also the proof of Lemma 2.4) we have

$$h(\langle F_{\varepsilon_0}(n \div 1), F_{\varepsilon_0}(n \div 1) \rangle) \geq n \quad \text{for all } n.$$

Coming to the other assumption, let g be any provably total function of the theory $\mathbf{IS}_1 + \text{RFN}_{\Sigma_1}^\circ(\mathbf{PA})$. We must show that $n \mapsto g(h(\langle F_{\varepsilon_0}(n \div 1), F_{\varepsilon_0}(n \div 1) \rangle))$ is eventually dominated by the function $n \mapsto \sigma(n+2, 10^{35n^2})$. To see this one applies Corollary 2.7 to the function $m \mapsto g(h(\langle m, m \rangle))$, which is still provably total in the theory $\mathbf{IS}_1 + \text{RFN}_{\Sigma_1}^\circ(\mathbf{PA})$. Having verified the assumptions Proposition 1.3 gives us a bound K' such that we have

$$p' > h(\langle F_{\varepsilon_0}(n \div 1), F_{\varepsilon_0}(n \div 1) \rangle)$$

whenever p' is a slow **PA**-proof of $\exists_y \psi(\bar{n}, y)$ with $n \geq K'$. To deduce the claim of (a), let p be a slow **PA**-proof of $\exists_N \text{PH}(\overline{10^{35n^2}}, \overline{n+3}, \overline{n+2}, N)$, still with $n \geq K'$. As we have seen above, this implies that there is a slow **PA**-proof of $\exists_y \psi(\bar{n}, y)$ below $h(p)$. By the bound that we have just established we must have

$$h(p) > h(\langle F_{\varepsilon_0}(n \div 1), F_{\varepsilon_0}(n \div 1) \rangle).$$

Since h is monotone this does indeed imply $p > \langle F_{\varepsilon_0}(n \div 1), F_{\varepsilon_0}(n \div 1) \rangle$. \square

We can derive the central result of the paper, claim (1) from the introduction:

Theorem 2.9. *The following holds:*

- (a) *There is a number K such that for all $n \geq K$ no proof of the formula $\exists_N \text{PH}(\overline{10^{35(n-2)^2}}, \overline{n+1}, \bar{n}, N)$ in the theory $\mathbf{IS}_{n \div 2}$ can have Gödel number less than or equal to $F_{\varepsilon_0}(n \div 3)$.*
- (b) *There is a number K such that for all $n \geq K$ no proof of the formula $\exists_N \text{PH}(\bar{8}, \overline{n+1}, \bar{n}, N)$ in the theory $\mathbf{IS}_{n \div 3}$ can have Gödel number less than or equal to $F_{\varepsilon_0}(n \div 4)$.*

Proof. We only write out the proof for (a), the proof of (b) being completely parallel: Let K' be the bound from Lemma 2.8, and set $K := \max\{K' + 2, 3\}$. Consider an arbitrary $n \geq K$ and a proof q of $\exists_N \text{PH}(\overline{10^{35(n-2)^2}}, \overline{n+1}, \bar{n}, N)$ in

the theory \mathbf{IS}_{n-2} . It follows that the pair $\langle q, F_{\varepsilon_0}(n-3) \rangle$ is a slow **PA**-proof of $\exists_N \text{PH}(\overline{10^{35(n-2)^2}}, \overline{n+1}, \overline{n}, N)$. Lemma 2.8 yields

$$\langle q, F_{\varepsilon_0}(n-3) \rangle > \langle F_{\varepsilon_0}(n-3), F_{\varepsilon_0}(n-3) \rangle.$$

Since the Cantor pairing is monotone we can conclude $q > F_{\varepsilon_0}(n-3)$, as desired. \square

By claim (2) from the introduction both $\exists_N \text{PH}(\overline{10^{35(n-2)^2}}, \overline{n+1}, \overline{n}, N)$ and the formula $\exists_N \text{PH}(\overline{8}, \overline{n+1}, \overline{n}, N)$ have short proofs in \mathbf{IS}_{n-1} . The fragment \mathbf{IS}_{n-2} in part (a) of the theorem is thus optimal. Concerning (b), it is currently open whether $\exists_N \text{PH}(\overline{8}, \overline{n+1}, \overline{n}, N)$ has a short proof in \mathbf{IS}_{n-2} . The crucial point is Lemma 2.6(b): If we could show that, for any fixed k and sufficiently large n , we have $F_{\omega_n+k}(n) \leq \sigma(n+2, 8)$ then we could replace \mathbf{IS}_{n-3} by the stronger fragment \mathbf{IS}_{n-2} in Theorem 2.9(b). The improved bound is not much stronger than [KS81, Theorem 3.10], which gives $F_{\omega_n}(n-1) \leq \sigma(n+2, 8)$. On the other hand we know $\sigma(n+2, 8) \leq F_{\omega_{13}}(4)$ from [KS81, Theorem 6.8], which means that the known upper and lower bounds already match very well.

3. THE PROVABLY TOTAL FUNCTIONS OF SLOW REFLECTION

The goal of this section is to provide a proof of Proposition 2.2, which we have postponed so far. We will need the following characterization of uniform Σ_1 -reflection over the fragments of Peano arithmetic:

Proposition 3.1. *We have*

$$\mathbf{IS}_1 \vdash \forall x (F_{\omega_x} \downarrow \leftrightarrow \text{RFN}_{\Sigma_1}(\mathbf{IS}_x)).$$

Proof. It is known that the equivalence $F_{\omega_n} \downarrow \leftrightarrow \text{RFN}_{\Sigma_1}(\mathbf{IS}_n)$ for fixed n is provable in \mathbf{IS}_1 (and in weaker theories): A model-theoretic proof can be found in [Par80] or [Som95, Proposition 6.8]. For a proof-theoretic approach (via iterated reflection principles) we refer to [Bek03, Theorem 1, Proposition 7.3, Remark 7.4]. The author has found no fully explicit argument that the formalization is uniform in n . We provide a detailed proof of this fact in [Fre15]: This is a proof-theoretic argument, formalizing the infinitary proof system from [BW87] by the method of [Buc91]. \square

Using this result and (5) we can view slow reflection as a statement about the fast-growing hierarchy of functions:

Corollary 3.2. *We have*

$$\mathbf{IS}_1 \vdash \text{RFN}_{\Sigma_1}^\circ(\mathbf{PA}) \leftrightarrow \forall x (F_{\varepsilon_0}(x) \downarrow \rightarrow F_{\omega_{x+1}} \downarrow).$$

Note that the “index shift”, stemming from the definition of slow proof, is indeed optimal: In view of $F_{\varepsilon_0}(x) \simeq F_{\omega_{x+1}}(x)$ we can deduce

$$\mathbf{IS}_1 \vdash \forall x (F_{\varepsilon_0}(x) \downarrow \rightarrow F_{\omega_{x+2}} \downarrow) \rightarrow \forall y F_{\varepsilon_0}(y) \downarrow$$

by induction on y . Thus a stronger slow reflection statement would collapse into the usual notion of Σ_1 -reflection over Peano Arithmetics. This explains why our bounds on proof size are relatively sharp.

Our next goal is to transform the Π_2 -statement $\forall x (F_{\varepsilon_0}(x) \downarrow \rightarrow F_{\omega_{x+1}} \downarrow)$ into a formula which defines a unary function.

Definition 3.3. The inverse $F_{\varepsilon_0}^{-1}$ of the function F_{ε_0} (see [FRW13, Definition 3.2]) is given by

$$F_{\varepsilon_0}^{-1}(x) := \max(\{z \leq x \mid \exists w \leq x F_{\varepsilon_0}(z) = w\} \cup \{0\}).$$

Note that the Δ_0 -definition of F_{ε_0} yields a Δ_0 -definition of $F_{\varepsilon_0}^{-1}$. To define a slow variant $F_{\varepsilon_0}^\diamond$ of the function F_{ε_0} we set

$$F_{\varepsilon_0}^\diamond(x) := F_{\omega_{F_{\varepsilon_0}^{-1}(x)+1}}(x),$$

which has the Σ_1 -definition

$$F_{\varepsilon_0}^\diamond(x) = y \iff \exists z(z = F_{\varepsilon_0}^{-1}(x) \wedge \exists \alpha(\alpha = \omega_{z+1} \wedge F_\alpha(x) = y)).$$

Clearly, z is bounded by x . In view of (4) the code of α is bounded by a polynomial in x . Thus the given definition of $F_{\varepsilon_0}^\diamond$ is Δ_0 in $\mathbf{I}\Sigma_1$.

Recall that the usual Σ_1 -reflection principle over Peano Arithmetic is equivalent to the totality of the function F_{ε_0} . We can now give a similar characterization of slow reflection:

Proposition 3.4. *We have*

$$\mathbf{I}\Sigma_1 \vdash \text{RFN}_{\Sigma_1}^\diamond(\mathbf{PA}) \leftrightarrow F_{\varepsilon_0}^\diamond \downarrow.$$

Proof. By Corollary 3.2 the claim of the proposition is equivalent to

$$\mathbf{I}\Sigma_1 \vdash \forall x(F_{\varepsilon_0}(x) \downarrow \rightarrow F_{\omega_{x+1}} \downarrow) \leftrightarrow F_{\varepsilon_0}^\diamond \downarrow.$$

To show the direction “ \rightarrow ” we work in $\mathbf{I}\Sigma_1$ and assume that the formula $\forall x(F_{\varepsilon_0}(x) \downarrow \rightarrow F_{\omega_{x+1}} \downarrow)$ holds. We have to prove $F_{\varepsilon_0}^\diamond(x) \downarrow$ for an arbitrary x . The finitely many $x < F_{\varepsilon_0}(0)$ are treated by Σ_1 -completeness. For $x \geq F_{\varepsilon_0}(0)$ the set $\{z \leq x \mid \exists w \leq x F_{\varepsilon_0}(z) = w\}$ is non-empty, so $F_{\varepsilon_0}^{-1}(x) =: z$ is an element of this set. In particular it follows that $F_{\varepsilon_0}(z)$ is defined. Then the assumption $\forall x(F_{\varepsilon_0}(x) \downarrow \rightarrow F_{\omega_{x+1}} \downarrow)$ tells us that $F_{\omega_{z+1}}$ is total. Thus $F_{\omega_{z+1}}(x)$ is defined, as required for $F_{\varepsilon_0}^\diamond(x) \downarrow$.

For the direction “ \leftarrow ”, assume that the function $F_{\varepsilon_0}^\diamond$ is total, let x be arbitrary, and assume that $F_{\varepsilon_0}(x)$ is defined. We have to prove that $F_{\omega_{x+1}}$ is total. By [FRW13, Lemma 2.3] it suffices to show that $F_{\omega_{x+1}}(y)$ is defined for arbitrarily large y . Since $F_{\varepsilon_0}(x)$ was assumed to be defined, we may consider an arbitrary y above this value. Then we have $x \leq F_{\varepsilon_0}^{-1}(y) =: z$. Invoking the totality of $F_{\varepsilon_0}^\diamond$ we learn that $F_{\varepsilon_0}^\diamond(y) = F_{\omega_{z+1}}(y)$ is defined. It follows by [FRW13, Lemma 2.4, Proposition 2.12, Lemma 2.3] that $F_{\omega_{x+1}}(y)$ is defined (and has value at most $F_{\varepsilon_0}^\diamond(y)$). \square

By the parenthesis at the end of the proof, the function $F_{\varepsilon_0}^\diamond$ dominates $F_{\omega_{x+1}}$ for values above $F_{\varepsilon_0}(x)$. In other words, $F_{\varepsilon_0}^\diamond$ eventually dominates any provably total function of Peano Arithmetic. In particular we have

$$\mathbf{PA} \not\vdash \text{RFN}_{\Sigma_1}^\diamond(\mathbf{PA}).$$

Since slow reflection implies slow consistency this was already known by [FRW13, Proposition 3.3]. It is important that the argument we just gave does not formalize in $\mathbf{I}\Sigma_1$ (or in fact Peano Arithmetic): To show that $F_{\varepsilon_0}^\diamond$ dominates $F_{\omega_{x+1}}$ we had to know that $F_{\varepsilon_0}(x)$ is defined. If this was different then $F_{\varepsilon_0}^\diamond \downarrow$ would imply $F_{\varepsilon_0} \downarrow$, contradicting the result that we are about to prove.

Recall that our goal is to bound the provably total functions of the theory $\mathbf{I}\Sigma_1 + \text{RFN}_{\Sigma_1}^\diamond(\mathbf{PA})$, or equivalently those of $\mathbf{I}\Sigma_1 + F_{\varepsilon_0}^\diamond \downarrow$. It is a classical result of Parsons in [Par70] that all provably total functions of $\mathbf{I}\Sigma_1$ are primitive recursive, and thus

dominated by F_k for some $k < \omega$. This is easily adapted to extensions by suitable base functions. We begin by building a hierarchy of ω stages on top of $F_{\varepsilon_0}^\diamond$:

Definition 3.5. By induction on k we define functions $F_{\varepsilon_0+k}^\diamond$: Set

$$\begin{aligned} F_{\varepsilon_0+0}^\diamond(n) &:= F_{\varepsilon_0}^\diamond(n), \\ F_{\varepsilon_0+k+1}^\diamond(n) &:= (F_{\varepsilon_0+k}^\diamond)^{n+1}(n), \end{aligned}$$

where the superscript $n+1$ denotes the number of iterations.

The fact that $F_{\varepsilon_0}^\diamond$ is a “suitable” base function for a primitive recursive hierarchy is expressed by the following result:

Lemma 3.6. *The following holds:*

$$\begin{aligned} n &< F_{\varepsilon_0+k}^\diamond(n), \\ n \leq m &\rightarrow F_{\varepsilon_0+k}^\diamond(n) \leq F_{\varepsilon_0+k}^\diamond(m), \\ k \leq k' &\rightarrow F_{\varepsilon_0+k}^\diamond(n) \leq F_{\varepsilon_0+k'}^\diamond(n), \\ n^2 &< F_{\varepsilon_0+k}^\diamond(n), \\ n \geq 1 &\rightarrow F_{\varepsilon_0+k}^\diamond(n) \geq 2. \end{aligned}$$

Proof. The first claim is shown by induction on k , with [Som95, Proposition 5.4] providing the base case. The second claim is again an induction on k : Concerning the base case, it is easy to see that $n \leq m$ implies $F_{\varepsilon_0}^{-1}(n) \leq F_{\varepsilon_0}^{-1}(m)$, and then

$$F_{\varepsilon_0}^\diamond(n) = F_{\omega_{F_{\varepsilon_0}^{-1}(n)+1}}(n) \leq F_{\omega_{F_{\varepsilon_0}^{-1}(m)+1}}(n) \leq F_{\omega_{F_{\varepsilon_0}^{-1}(m)+1}}(m) = F_{\varepsilon_0}^\diamond(m)$$

follows from [FRW13, Lemma 2.3, Proposition 2.12]. The induction step holds by

$$F_{\varepsilon_0+k+1}^\diamond(n) = (F_{\varepsilon_0+k}^\diamond)^{n+1}(n) \leq (F_{\varepsilon_0+k}^\diamond)^{n+1}(m) \leq (F_{\varepsilon_0+k}^\diamond)^{m+1}(m) = F_{\varepsilon_0+k+1}^\diamond(m),$$

due to the induction hypothesis and the first claim. The third claim is easily proved by induction on $k' \geq k$, using the first claim. This reduces the fourth claim to the case $k = 0$. There [Som95, Proposition 5.4] tells us that $\omega_{F_{\varepsilon_0}^{-1}(n)+1} \geq 2$ implies

$$F_{\varepsilon_0}^\diamond(n) = F_{\omega_{F_{\varepsilon_0}^{-1}(n)+1}}(n) > n^2.$$

Finally, the fifth claim follows from the first. \square

We can now bound the provably total functions of slow reflection:

Proposition 3.7. *Assume that $g : \mathbb{N} \rightarrow \mathbb{N}$ is provably total in $\mathbf{IS}_1 + \mathbf{RFN}_{\Sigma_1}^\diamond(\mathbf{PA})$. Then there is a number k such that we have $g(n) \leq F_{\varepsilon_0+k}^\diamond(n)$ for all $n \geq 1$.*

Proof. Proposition 3.4 allows us to replace $\mathbf{IS}_1 + \mathbf{RFN}_{\Sigma_1}^\diamond(\mathbf{PA})$ by $\mathbf{IS}_1 + F_{\varepsilon_0}^\diamond \downarrow$. Recall that the formula $F_{\varepsilon_0}^\diamond(x) = y$ is Δ_0 in \mathbf{IS}_1 . We assume that this Δ_0 -definition is used in the axiom $F_{\varepsilon_0}^\diamond \downarrow$. Now the claim is a standard application of partial cut elimination. Let us provide some details:

First, we formulate the theory $\mathbf{IS}_1 + F_{\varepsilon_0}^\diamond \downarrow$ in the sequent calculus of [Bus98, 2.3.2, 2.4.6]. As initial sequents we have $\psi \Rightarrow \psi$ for any atomic formula ψ . Additionally, any Π_2 -axiom $\forall \vec{x} \exists \vec{y} \varphi(x, y)$ of the theory $\mathbf{IS}_1 + F_{\varepsilon_0}^\diamond \downarrow$ (including the equality axioms but excluding induction) gives rise to the initial sequents

$$\Rightarrow \exists \vec{y} \varphi(\vec{t}, \vec{y})$$

with arbitrary terms \vec{t} . To implement induction for the Σ_1 -formula $\varphi(x)$, possibly containing other free variables, we allow the rules

$$\frac{\varphi(x), \Gamma \Rightarrow \Delta, \varphi(x+1)}{\varphi(0), \Gamma \Rightarrow \Delta, \varphi(t)} \quad (x \text{ not free in } \Gamma, \Delta)$$

where t is an arbitrary term. Assume that Γ and Δ contain only Σ_1 -formulas and that $\Gamma \Rightarrow \Delta$ is provable in the sequent calculus version of $\mathbf{IS}_1 + F_{\varepsilon_0}^\circ \downarrow$. Then the partial cut elimination procedure of [Bus98, Corollary 2.4.7] produces a proof in which only Σ_1 -formulas occur.

Next, let us introduce some notation to track bounds through a free-cut free sequent calculus proof: Consider a Σ_1 -formula $\varphi \equiv \exists_{\vec{y}} \psi(\vec{x}, \vec{y})$, an assignment e of values to the variables \vec{x} , and a number n . We write $\text{True}(\varphi; e, n)$ to express that there are numbers $\vec{m} \leq n$ such that the Δ_0 -formula $\psi(\vec{x}, \vec{m})$ is true under the variable assignment e . Now consider a sequent $\Gamma \Rightarrow \Delta$ which only contains Σ_1 -formulas. Then $\text{True}(\Gamma \Rightarrow \Delta; n, m)$ abbreviates the following statement: For any e which assigns values below n to the free variables of $\Gamma \Rightarrow \Delta$, if $\text{True}(\varphi; e, n)$ holds for all $\varphi \in \Gamma$ then $\text{True}(\psi; e, m)$ holds for some $\psi \in \Delta$.

We will prove the following statement:

- (9) Consider a sequent calculus proof of $\Gamma \Rightarrow \Delta$ in the theory $\mathbf{IS}_1 + F_{\varepsilon_0}^\circ \downarrow$, such that only Σ_1 -formulas appear in the proof. If k bounds the height of the proof tree and l bounds the maximal depth of a term that appears in the proof, then we have $\text{True}(\Gamma \Rightarrow \Delta; n, F_{\varepsilon_0+2k+l+1}^\circ(n))$ for all $n \geq 1$.

Before we prove this statement, let us show how the claim of the proposition can be deduced: By assumption we have a Σ_1 -formula $g(x) = y$ which defines the graph of g , and such that the sequent $\Rightarrow \exists_y g(x) = y$ is provable in (the sequent calculus version of) the theory $\mathbf{IS}_1 + F_{\varepsilon_0}^\circ \downarrow$. By partial cut-elimination we may assume that this proof contains only Σ_1 -formulas. Then (9) provides a number k such that we have $\text{True}(\Rightarrow \exists_y g(x) = y; n, F_{\varepsilon_0+k}^\circ(n))$ for all $n \geq 1$. Let e_n assign the value $n \geq 1$ to x . We can conclude $\text{True}(\exists_y g(x) = y; e_n, F_{\varepsilon_0+k}^\circ(n))$. This implies that $\exists_y g(n) = y$ is true for some $y \leq F_{\varepsilon_0+k}^\circ(n)$, i.e. that we have $g(n) \leq F_{\varepsilon_0+k}^\circ(n)$.

To prepare the proof of (9) we need the following auxiliary result:

Let t be a term of depth l . If e assigns values below $n \geq 1$ to the free variables of t then the value $\text{Val}(t; e)$ of t under e is bounded by $F_{\varepsilon_0+l}^\circ(n)$.

This is shown by a straightforward induction on t , using Lemma 3.6. Note that terms are built from zero and variables by function symbols for successor, addition and multiplication. Building on this, one establishes (9) by induction over the assumed sequent calculus proof of $\Gamma \Rightarrow \Delta$. We only look at three cases: The most interesting initial sequent is an axiom of the form $\Rightarrow \exists_y F_{\varepsilon_0}^\circ(t) = y$, where the depth of t is bounded by l . Assuming that e assigns values below $n \geq 1$ to the free variables of t , it is enough to observe

$$F_{\varepsilon_0}^\circ(\text{Val}(t; e)) \leq F_{\varepsilon_0}^\circ(F_{\varepsilon_0+l}^\circ(n)) \leq (F_{\varepsilon_0+l}^\circ)^{n+1}(n) = F_{\varepsilon_0+l+1}^\circ(n).$$

Next, consider a rule

$$\frac{\Gamma \Rightarrow \Delta, x \leq t \rightarrow \varphi(x)}{\Gamma \Rightarrow \Delta, \forall_{x \leq t} \varphi(x)} \quad (x \text{ not free in } \Gamma, \Delta \text{ or } t)$$

which introduces a bounded universal quantifier on the right. Assume that $k+1$ bounds the height of the proof tree which ends in the lower sequent, and that l bounds the depth of terms occurring in that proof tree. To verify that we have $\text{True}(\Gamma \Rightarrow \Delta, \forall_{x \leq t} \varphi(x); n, F_{\varepsilon_0+2(k+1)+l+1}^\circ(n))$, let e assign values below n to the

free variables of $\Gamma \Rightarrow \Delta, \forall_{x \leq t} \varphi(x)$, and assume that $\text{True}(\psi; e, n)$ holds for all $\psi \in \Gamma$. Also assume that $\text{True}(\rho; e, F_{\varepsilon_0+2(k+1)+l+1}^\diamond(n))$ fails for all $\rho \in \Delta$. We need to show that $\forall_{x \leq t} \varphi(x)$ is true under the assignment e (independent of a bound, since we are concerned with a Δ_0 -formula). It suffices to prove that $x \leq t \rightarrow \varphi(x)$ is true under any assignment e' which extends e by sending x to a value below $\text{Val}(t; e) \leq F_{\varepsilon_0+l}^\diamond(n) =: N$. Now the induction hypothesis yields

$$\text{True}(\Gamma \Rightarrow \Delta, x \leq t \rightarrow \varphi(x); N, F_{\varepsilon_0+2k+l+1}^\diamond(N)).$$

Observe $n \leq N$ and

$$F_{\varepsilon_0+2k+l+1}^\diamond(N) \leq (F_{\varepsilon_0+2k+l+1}^\diamond)^{n+1}(n) = F_{\varepsilon_0+2(k+1)+l}^\diamond(n) \leq F_{\varepsilon_0+2(k+1)+l+1}^\diamond(n).$$

Thus we still have $\text{True}(\psi; e', N)$ for all $\psi \in \Gamma$, and $\text{True}(\rho; e', F_{\varepsilon_0+2k+l+1}^\diamond(N))$ still fails for all $\rho \in \Delta$. We can conclude $\text{True}(x \leq t \rightarrow \varphi(x); e', F_{\varepsilon_0+2k+l+1}^\diamond(N))$, meaning that $x \leq t \rightarrow \varphi(x)$ is indeed true under the variable assignment e' .

Finally, consider the case of an induction rule

$$\frac{\varphi(x), \Gamma \Rightarrow \Delta, \varphi(x+1)}{\varphi(0), \Gamma \Rightarrow \Delta, \varphi(t)} \quad (x \text{ not free in } \Gamma, \Delta).$$

We may assume that x does not occur in t . Suppose that $k+1$ bounds the height of the proof tree which ends in the lower sequent, and that l bounds the depth of terms occurring in it. To check $\text{True}(\varphi(0), \Gamma \Rightarrow \Delta, \varphi(t); n, F_{\varepsilon_0+2(k+1)+l+1}^\diamond(n))$, let e assign values below n to the free variables of $\varphi(0), \Gamma \Rightarrow \Delta, \varphi(t)$. Assume that we have $\text{True}(\varphi(0); e, n)$, and that $\text{True}(\psi; e, n)$ holds for all $\psi \in \Gamma$. We also assume that $\text{True}(\rho; e, F_{\varepsilon_0+2(k+1)+l+1}^\diamond(n))$ fails for all $\rho \in \Delta$. We need to establish $\text{True}(\varphi(t); e, F_{\varepsilon_0+2(k+1)+l+1}^\diamond(n))$. Let us reduce this to the following claim:

$$(10) \quad \text{Let } e_m \text{ extend } e \text{ by assigning the value } m \text{ to the variable } x. \text{ Set } N_m := (F_{\varepsilon_0+2k+l+1}^\diamond)^m(n). \text{ Then we have } \text{True}(\varphi(x); e_m, N_m) \text{ for all } m \leq N_1.$$

In view of $\text{Val}(t; e) \leq N_1$ this claim gives $\text{True}(\varphi(x); e_{\text{Val}(t; e)}, N_{\text{Val}(t; e)})$, or equivalently $\text{True}(\varphi(t); e, N_{\text{Val}(t; e)})$. To get $\text{True}(\varphi(t); e, F_{\varepsilon_0+2(k+1)+l+1}^\diamond(n))$ one observes

$$\begin{aligned} N_{\text{Val}(t; e)} &= (F_{\varepsilon_0+2k+l+1}^\diamond)^{\text{Val}(t; e)}(n) \leq (F_{\varepsilon_0+2k+l+1}^\diamond)^{N_1+1}(N_1) = \\ &= F_{\varepsilon_0+2k+l+2}^\diamond(N_1) \leq (F_{\varepsilon_0+2k+l+2}^\diamond)^2(n) \leq F_{\varepsilon_0+2(k+1)+l+1}^\diamond(n). \end{aligned}$$

It remains to show (10), which we do by induction on $m \leq N_1$: For $m = 0$ it suffices to invoke $\text{True}(\varphi(0); e, n)$ from above. Concerning the step, the induction hypothesis of (9) provides

$$\text{True}(\varphi(x), \Gamma \Rightarrow \Delta, \varphi(x+1); N_m, F_{\varepsilon_0+2k+l+1}^\diamond(N_m)).$$

Since x is not free in Γ or Δ , and in view of $n \leq N_m$ and

$$\begin{aligned} F_{\varepsilon_0+2k+l+1}^\diamond(N_m) &= (F_{\varepsilon_0+2k+l+1}^\diamond)^{m+1}(n) \leq \\ &\leq (F_{\varepsilon_0+2k+l+1}^\diamond)^{N_1+1}(N_1) \leq F_{\varepsilon_0+2(k+1)+l+1}^\diamond(n), \end{aligned}$$

we still have $\text{True}(\psi; e_m, N_m)$ for all $\psi \in \Gamma$, and $\text{True}(\rho; e_m, F_{\varepsilon_0+2k+l+1}^\diamond(N_m))$ fails for all $\rho \in \Delta$. The induction hypothesis of (10) gives $\text{True}(\varphi(x); e_m, N_m)$. We can conclude $\text{True}(\varphi(x+1); e_m, F_{\varepsilon_0+2k+l+1}^\diamond(N_m))$. This is equivalent to the desired $\text{True}(\varphi(x); e_{m+1}, N_{m+1})$. \square

To complement this result we bound the functions $F_{\varepsilon_0+k}^\diamond$. This could be described as the combinatorial core of Proposition 2.2.

Lemma 3.8. *Consider numbers k, l, m and n with*

$$\begin{aligned} m &\geq 2k+1, \\ n &\leq F_{\omega_m+m-k}(m), \\ l &\leq n+1. \end{aligned}$$

Then we have

$$(F_{\varepsilon_0+k}^\diamond)^l(n) \leq (F_{\omega_m+k})^l(n).$$

Proof. The proof is by induction on k , throughout which m may be fixed. The case $k=0$ is established by a side induction on l . The base $l=0$ amounts to the trivial $n \leq n$, and we come to the side induction step $l \rightsquigarrow l+1$: There we consider an arbitrary n such that the assumptions of the proposition are satisfied, i.e. such that we have $m \geq 1$, $n \leq F_{\omega_m+m}(m)$ and $l+1 \leq n+1$. We can use the side induction hypothesis to deduce

$$\begin{aligned} (F_{\varepsilon_0}^\diamond)^l(n) &\leq (F_{\omega_m})^l(n) < (F_{\omega_m})^{n+1}(n) = F_{\omega_m+1}(n) \leq \\ &\leq F_{\omega_m+1}(F_{\omega_m+m}(m)) \leq (F_{\omega_m+m})^2(m) \leq F_{\omega_m+m+1}(m) \leq F_{\varepsilon_0}(m). \end{aligned}$$

Conversely, this means that we have

$$m' := F_{\varepsilon_0}^{-1}((F_{\varepsilon_0}^\diamond)^l(n)) < m,$$

and by the definition of $F_{\varepsilon_0}^\diamond$ we get

$$(F_{\varepsilon_0}^\diamond)^{l+1}(n) = F_{\varepsilon_0}^\diamond((F_{\varepsilon_0}^\diamond)^l(n)) = F_{\omega_{m'}+1}((F_{\varepsilon_0}^\diamond)^l(n)) \leq F_{\omega_m}((F_{\varepsilon_0}^\diamond)^l(n)).$$

Another application of the side induction hypothesis yields

$$F_{\omega_m}((F_{\varepsilon_0}^\diamond)^l(n)) \leq F_{\omega_m}((F_{\omega_m})^l(n)) = (F_{\omega_m})^{l+1}(n).$$

Let us come to the step $k \rightsquigarrow k+1$ of the main induction. Note that here we can assume $m \geq 2k+3$, and thus $m-k-1 \geq k+2$. We have to show that the inequality

$$(11) \quad (F_{\varepsilon_0+k+1}^\diamond)^l(n) \leq (F_{\omega_m+k+1})^l(n)$$

holds for all numbers l and n such that we have $n \leq F_{\omega_m+m-k-1}(m)$ and $l \leq n+1$. This is again established by a side induction on l , in which the base case $l=0$ is again trivial. Coming to the step $l \rightsquigarrow l+1$, let us consider an arbitrary n with $n \leq F_{\omega_m+m-k-1}(m)$ and $l+1 \leq n+1$. The side induction hypothesis first gives

$$\begin{aligned} N := (F_{\varepsilon_0+k+1}^\diamond)^l(n) &\leq (F_{\omega_m+k+1})^l(n) \leq (F_{\omega_m+k+1})^{n+1}(n) = F_{\omega_m+k+2}(n) \leq \\ &\leq F_{\omega_m+k+2}(F_{\omega_m+m-k-1}(m)) \leq (F_{\omega_m+m-k-1})^2(m) \leq F_{\omega_m+m-k}(m). \end{aligned}$$

Thanks to these inequalities we can use the induction hypothesis of the main induction with N and $N+1$ at the place of n and l , respectively. We obtain

$$\begin{aligned} (F_{\varepsilon_0+k+1}^\diamond)^{l+1}(n) &= F_{\varepsilon_0+k+1}^\diamond(N) = (F_{\varepsilon_0+k}^\diamond)^{N+1}(N) \leq (F_{\omega_m+k})^{N+1}(N) = \\ &= F_{\omega_m+k+1}(N) \leq F_{\omega_m+k+1}((F_{\omega_m+k+1})^l(n)) = (F_{\omega_m+k+1})^{l+1}(n). \end{aligned}$$

This completes the step of the side induction for (11), and thus it completes the step of the main induction itself. \square

Putting pieces together, we can deduce the open result:

Proof of Proposition 2.2. Let g be provably total in $\mathbf{IS}_1 + \text{RFN}_{\Sigma_1}^\diamond(\mathbf{PA})$. Proposition 3.7 gives us a number k such that we have

$$g(n) \leq F_{\varepsilon_0+k}^\diamond(n) \quad \text{for all } n \geq 1.$$

Set $N := 2k + 1$ and consider an arbitrary $n \geq N$. We invoke Lemma 3.8 with n at the place of m , with $F_{\varepsilon_0}(n - 1)$ at the place of n , and with $l = 1$. Indeed we have

$$F_{\varepsilon_0}(n - 1) = F_{\omega_n}(n - 1) \leq F_{\omega_n+n-k}(n - 1) \leq F_{\omega_n+n-k}(n).$$

Thus the assumptions of Lemma 3.8 are satisfied, and the lemma yields

$$F_{\varepsilon_0+k}^\diamond(F_{\varepsilon_0}(n - 1)) \leq F_{\omega_n+k}(F_{\varepsilon_0}(n - 1)).$$

In view of $F_{\varepsilon_0}(n - 1) = F_{\omega_n}(n - 1) \leq F_{\omega_n}(n)$ we get

$$F_{\varepsilon_0+k}^\diamond(F_{\varepsilon_0}(n - 1)) \leq F_{\omega_n+k}(F_{\omega_n}(n)) \leq (F_{\omega_n+k})^2(n) \leq F_{\omega_n+k+1}(n).$$

Putting the inequalities together gives

$$g(F_{\varepsilon_0}(n - 1)) \leq F_{\varepsilon_0+k}^\diamond(F_{\varepsilon_0}(n - 1)) \leq F_{\omega_n+k+1}(n) \quad \text{for all } n \geq N,$$

as required for Proposition 2.2. \square

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